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Chapter 2

Tutorial Length
3 Hours 10 Mins

Linear Programming - Introduction

Topics Covered

Preliminary Concepts
- -How to graph a line
- -Graphing and inequalities
- -Points of Intersection

Linear Programming Concepts
- -Linear Programming definitions
- -Example to explain definitions
- -Examples of special problems
- -Word problems and program formulation

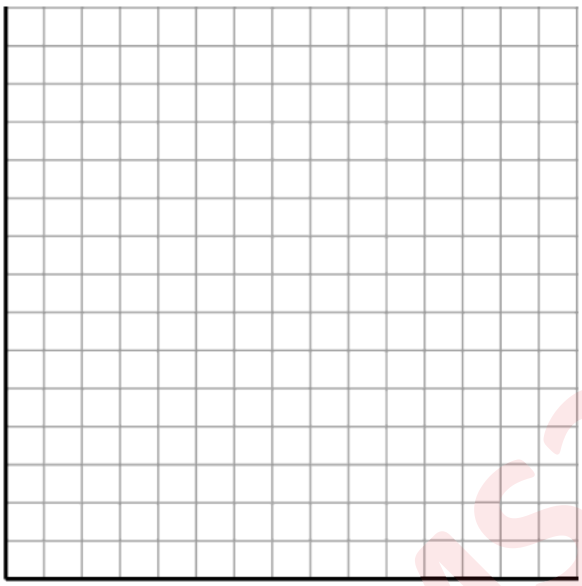
Graphing Basics

How To Graph

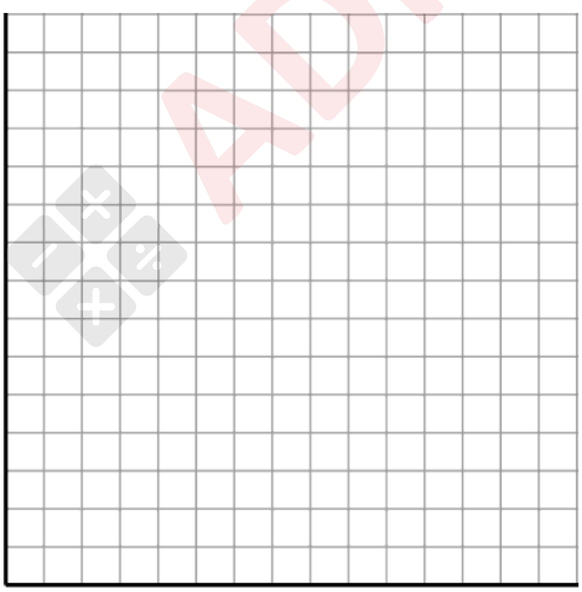
To graph a line, set one of the variables to 0, and solve for the other variable. This will create a coordinate point. Do this process again for the other variable to create a second coordinate point. Plot the coordinate points and connect with a straight line.

The first variable is along the horizontal axis, and the second variable is along the vertical axis. In general, we are interested only in the upper right quadrant when graphing.

Example Plot $3X + 2Y = 12$



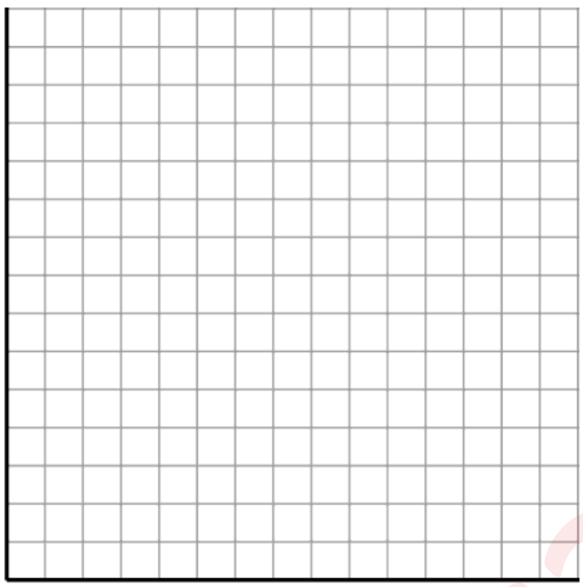
Example Plot $5P + 2D = 10$



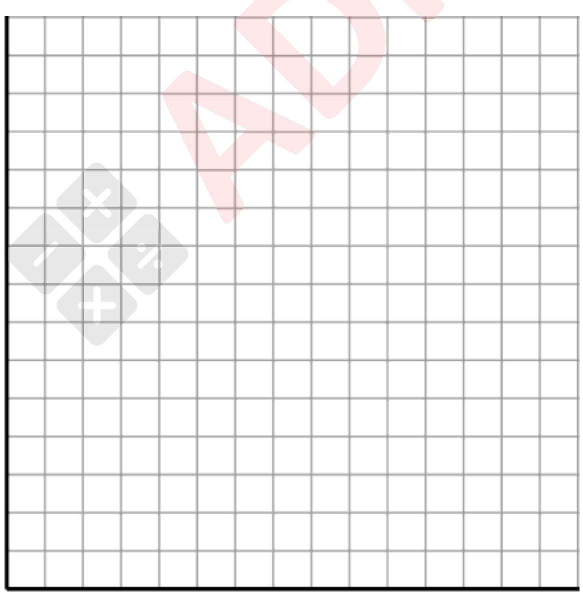
Graphing Basics

Even though we will only be interested in the quadrant with positive numbers, sometimes we will get points that are negative. In this case, extend the axis to help draw the line OR find a new coordinate point by plugging in a value other than 0 and re-solving the equation to find a new coordinate point.

Example Plot $4X - 2Y = -4$



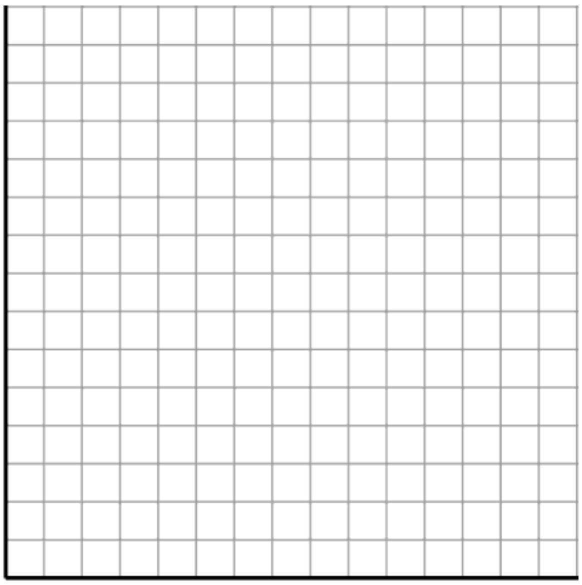
Example Plot $2V - 3P = 6$



Graphing Basics

Sometimes you will get the same coordinate point twice. In this scenario, plug in any value for one of the variables and solve for the other create a new coordinate point.

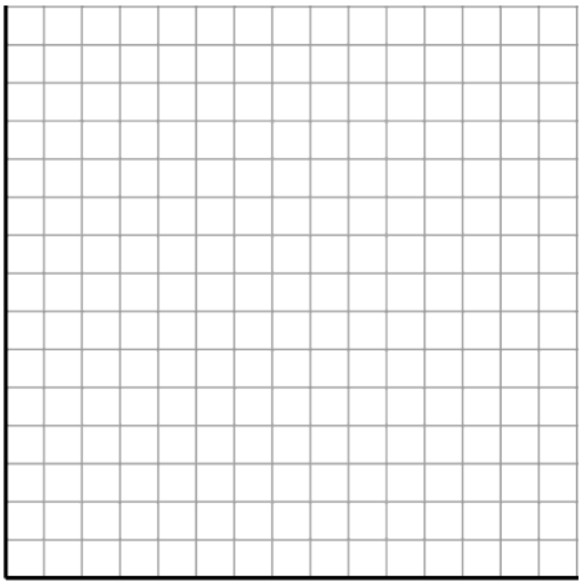
Example Plot $4X - 5Y = 0$



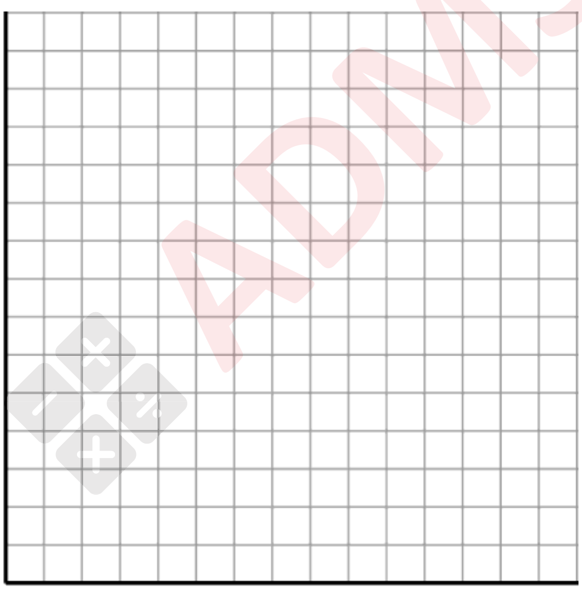
Graphing Basics

What happens when there is only one variable in the equation? In this case, it will be a straight line in the horizontal or vertical direction. X=# is a vertical line, Y=# is a horizontal line.

Example Plot $X = 5$



Example Plot $Y = 1$



Graphing and Inequalities

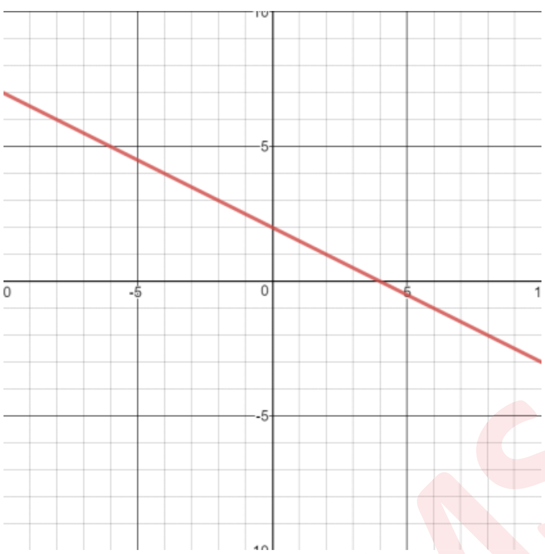
It will be important to know what region is being referred to once a line is plotted.

For a linear equation $aX + bY \geq c$, the region is above the line
For a linear equation $aX + bY \leq c$, the region is below the line

Important Note: The above is only true if b is a positive number. If b is negative, then the opposite is true! (The sign of a does not matter). So essentially, any time the <u>second variable</u> has a negative coefficient, it is the opposite region.

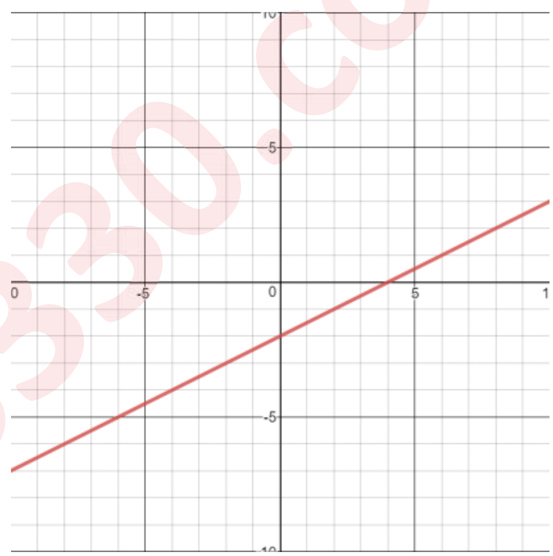
Example Consider the plot of $2X + 4Y = 8$

What region is indicated by:
A) $2X + 4Y \leq 8$
B) $2X + 4Y \geq 8$



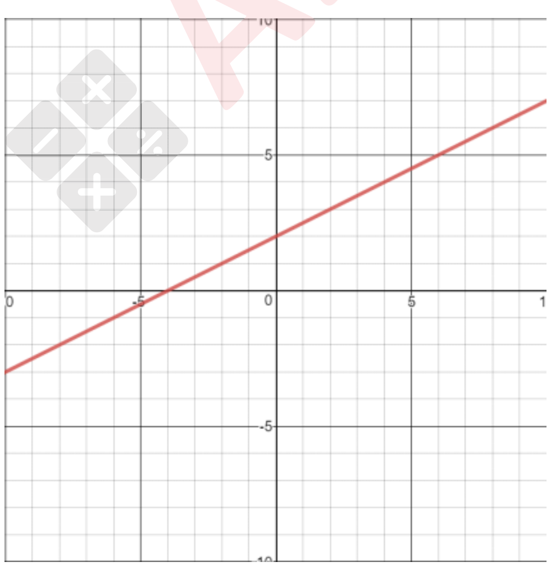
Example Consider the plot of $2X - 4Y = 8$

What region is indicated by:
A) $2X - 4Y \leq 8$
B) $2X - 4Y \geq 8$



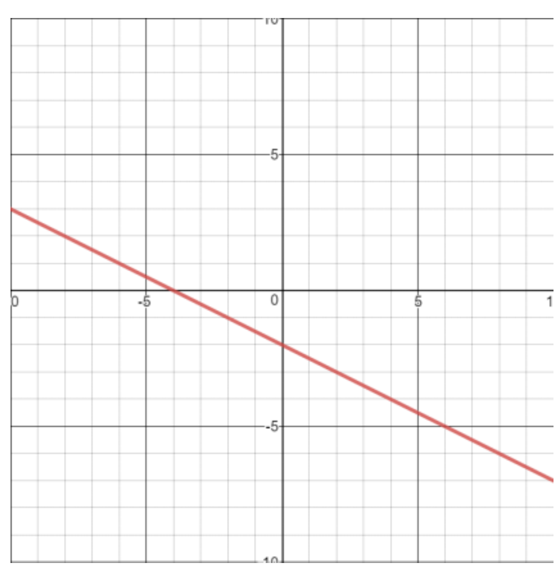
Example Consider the plot of $-2X + 4Y = 8$

What region is indicated by:
A) $-2X + 4Y \leq 8$
B) $-2X + 4Y \geq 8$



Example Consider the plot of $-2X - 4Y = 8$

What region is indicated by:
A) $-2X - 4Y \leq 8$
B) $-2X - 4Y \geq 8$



Graphing and Inequalities

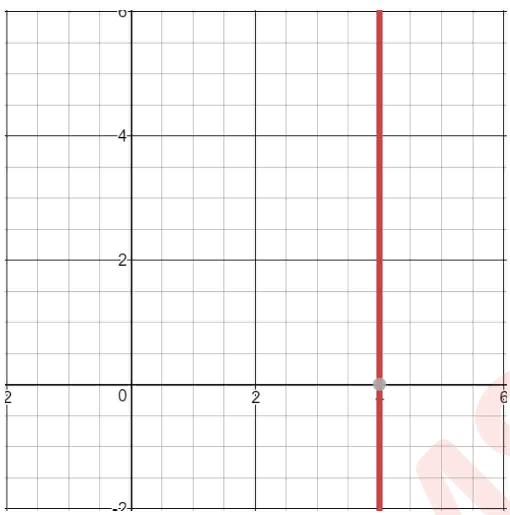
For a linear equation $X \geq C$, the region is to the right side of the line
For a linear equation $X \leq C$, the region is to the left side of the line

For a linear equation $Y \geq C$, the region is above the line
For a linear equation $Y \leq C$, the region is below the line

Example Consider the plot of $X = 4$

What region is indicated by:

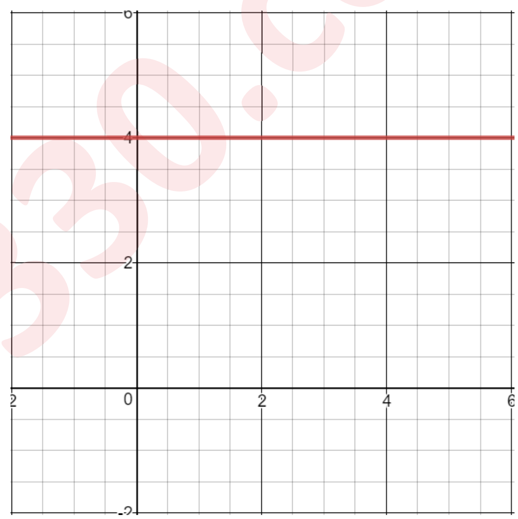
A) $X \leq 4$
B) $X \geq 4$



Example Consider the plot of $Y = 4$

What region is indicated by:

A) $Y \leq 4$
B) $Y \geq 4$



Points of Intersection

Determining Points of Intersection

We will have to graph multiple lines on a single plot and we will have to find points of intersection. There are multiple techniques one can use to solve for the point of intersection. We will discuss a few potential methods.

1) Elimination Method (Most Common Method)

Write the equations one on top of the other. The goal is to make the coefficient of one of the variables the same value so that the equations can be subtracted. (Alternatively, make the coefficients the same but opposite signs, so that the equations can be added) This will allow us to solve for the remaining variable. Once we solve for one variable, we can put that in to either equation to solve for the other variable. Multiply one or both equations by whatever number is required to make the coefficients the same (or same but opposite signs).

2) Substitution Method

Solve for one variable from either equation, substitute this in to the other equation, and solve for the remaining variable. Then substitute this calculated value to solve for the other variable.

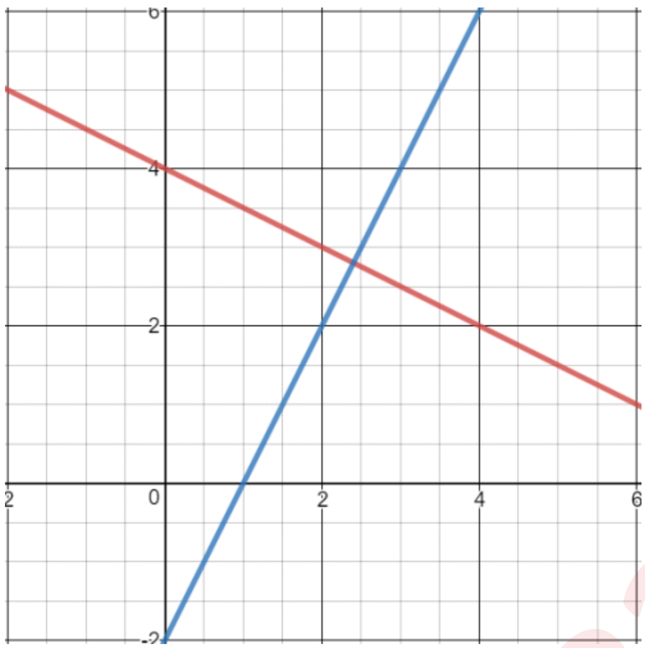
3) Calculator

Some calculators are capable of calculating the point of intersection for us.

Points of Intersection

Example

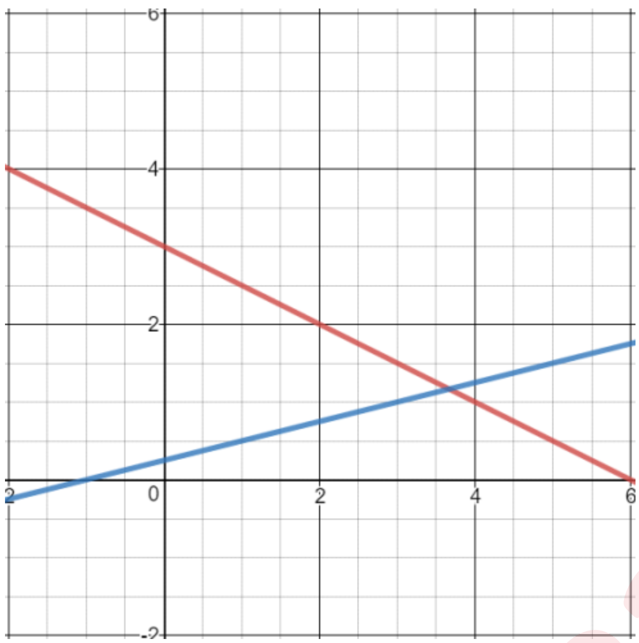
Consider the plot of $X + 2Y = 8$ (red) and $2X - Y = 2$ (blue).
What is the point of intersection?



Points of Intersection

Example

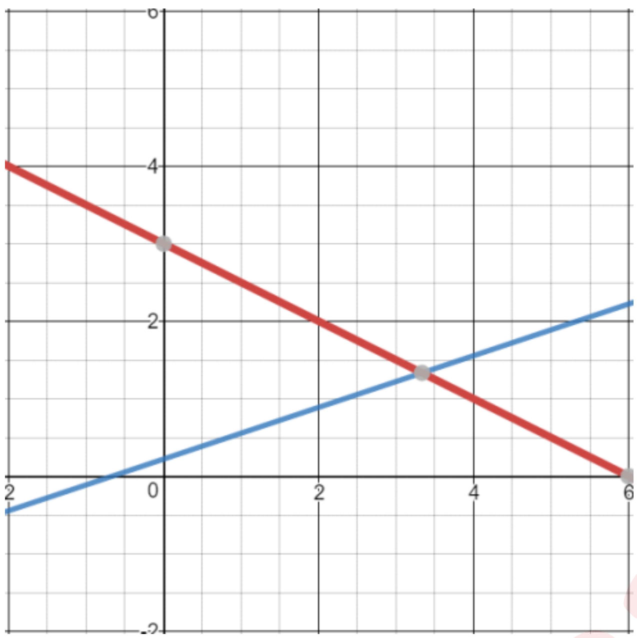
Consider the plot of $2X + 4Y = 12$ (red) and $-2X + 8Y = 2$ (blue).
What is the point of intersection?



Points of Intersection

Example

Consider the plot of $2X + 4Y = 12$ (red) and $-3X + 9Y = 2$ (blue).
What is the point of intersection?



Linear Programming

Introduction

The objective of linear programming is the process of finding an optimal solution to a problem. The optimal solution will require maximizing or minimizing a certain quantity. There will be restrictions (constraints) that will have to be taken in to account in while finding the optimal solution.

To formulate the linear program:

1. Define the decision variables (the controllable inputs)

2. Formulate the objective function (an equation that we are trying to maximize or minimize)

3. Describe the constraints
    -Variables are typically written on the left side of the constraint
    -Constrains can have $\leq, \geq,$ or $=$
    -There is always a non-negativity constraint that must be added as well

To solve the linear program:

1. Plot the constraint lines

2. Find the region where all constraints are satisfied simultaneously. Do this by eliminating regions that do not meet the constraint requirement. The region you are left with is known as the feasible region.

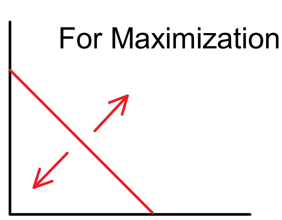
3. Solve the problem by either of the following methods:

    Method 1:
    Find all extreme points (corners/edges) of the feasible region and plug these values in to the objective function.

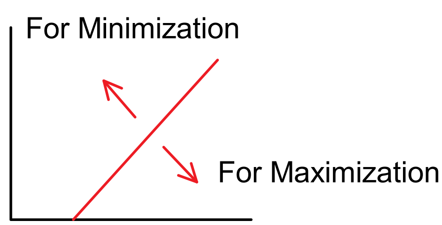
    For maximization, the points that produce the largest value are the answer.
    For minimization, the point that produce the smallest value are the answer.

    Method 2
    Plot the objective function after assigning it an arbitrary value.

    If it is a maximization objective, move the line in a direction that would increase the objective function value. If it is a minimization objective, move the line in a direction that would decrease the objective function value. The final extreme point (corner/edge) the line touches on the feasible region (without the line actually passing through the region) is the solution. The graphs below indicate which direction to move the line depending on the plot of the objective function. Generally speaking, move the line to the right for Maximization problems, and to the left for Minimization problems.



For Minimization



Terminology

Binding Constraints

The constraints from which the optimal solution is found

Nonbinding Constraints

The constraints that are not part of the optimal solution.

Redundant Constraint

A constraint that does not affect the feasible region and thus does not affect the optimal solution. Graphically, this is a constraint line that does not border the feasible region. If a constraint is determined to be redundant, it can be safely ignored.

Slack/Surplus

The slack/surplus is the difference between the RHS and LHS of a constraint when plugging in the optimal solution values.

Slack is the amount of unused material when we have a $\leq$ constraint.
This can be thought of as leftover material.

Surplus is the amount of overused material when we have $\geq$ constraint
This can be thought of as the amount used over some minimum requirement.

Binding constraints will always have a slack/surplus of 0

Standard Form

All the constraints are written with equalities (= sign's only)
For constraints that are $\leq$, we have to **add** S_i on the LHS
For constraints that are $\geq$, we have to **subtract** S_i on the LHS
The S_i variables have to be added to the objective function with coefficient 0
The S_i variables have to be added to the non-negativity constraint

Shadow Price

The difference between the optimal value of the objective function and the optimal value of the objective function when the RHS of a constraint is increased by one unit. The shadow price only needs to be calculated for binding constraints. For non-binding constraints, the shadow price is always equal to 0.

Shadow Price = New Optimal Value - Old Optimal Value

If shadow price is positive, it represents the amount by which the objective function increases for each unit change in the constraint.

If shadow price is negative, it represents the amount by which the objective function decreases for each unit change in the constraint.

Linear Programming Example

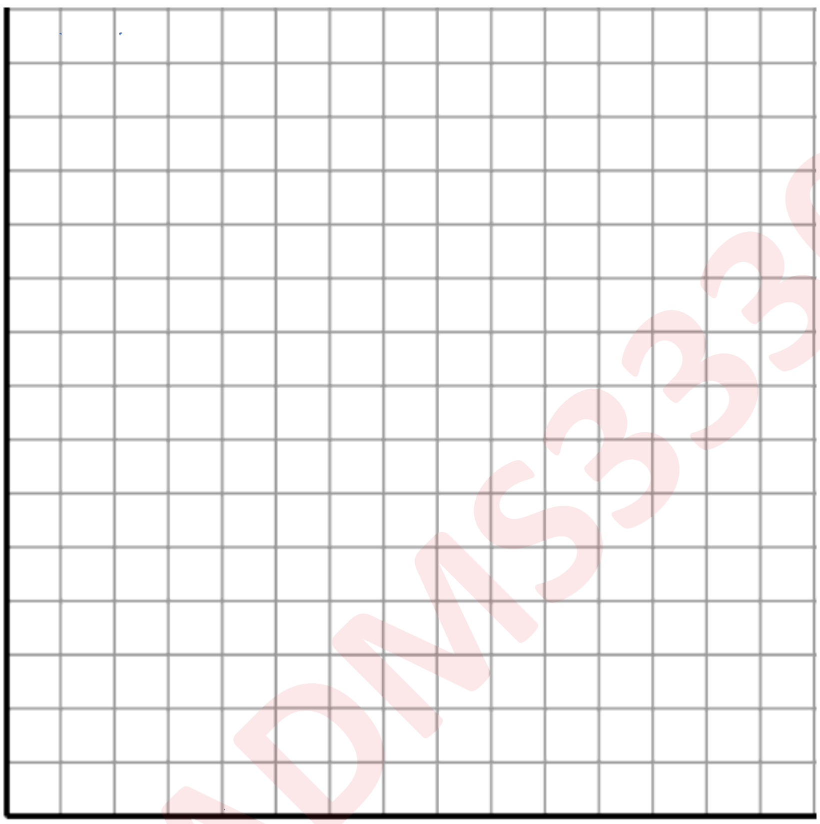
Example

Consider the following linear program.

MAX $2X + 3Y$
s.t.

$$X + 2Y \geq 12$$
$$X + Y \leq 10$$
$$X, Y \geq 0$$

A) Graph the feasible region.



B) What are the extreme points of the feasible region?

Linear Programming Example

C) Find the optimal solution.

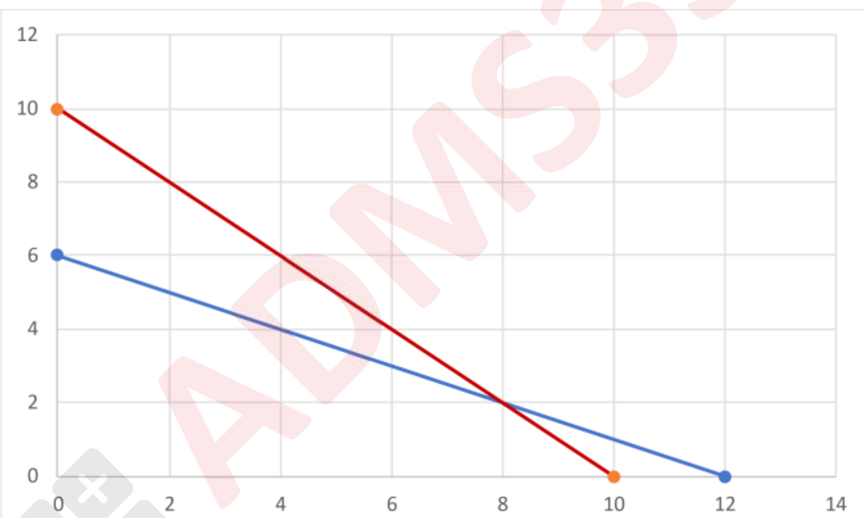
D) Write the linear program in standard form

E) Calculate the slack/surplus for each constraint.

F) What are the binding constraints?

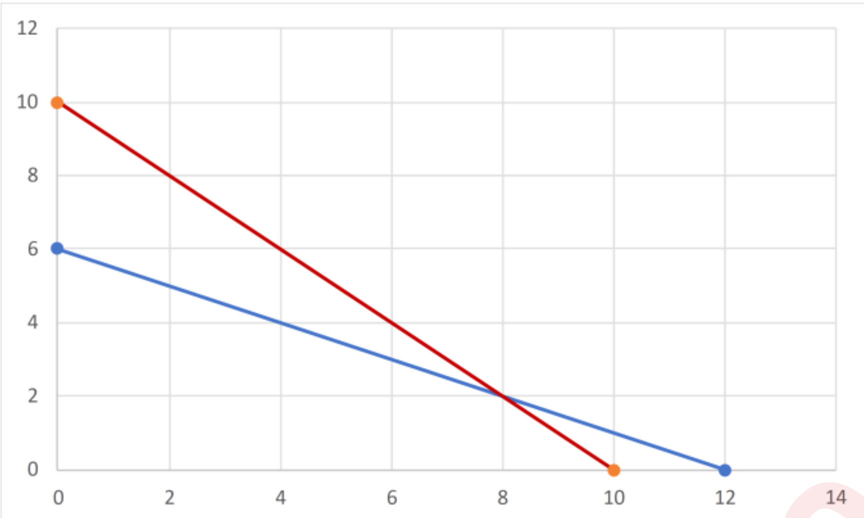
G) Calculate the shadow prices.

H) What if the objective was MIN $2X + 3Y$, what would be the optimal solution?

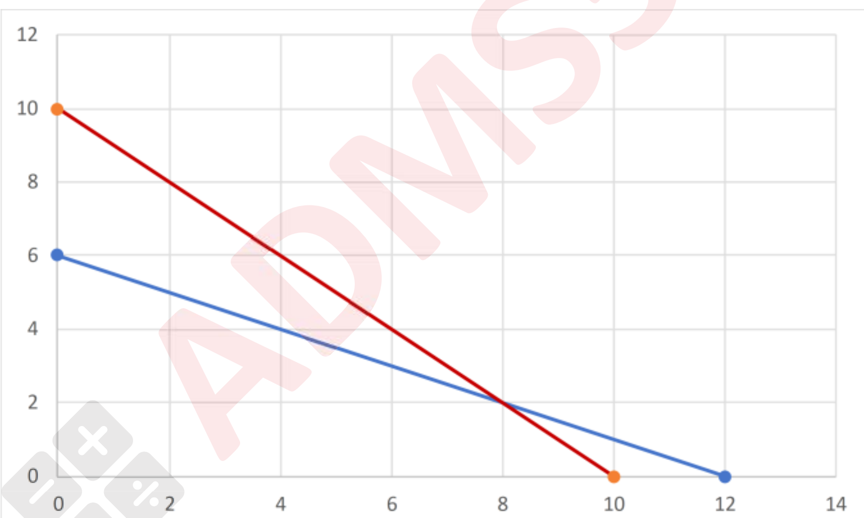


Linear Programming Example

I) What if the objective was MAX $2X - 3Y$, what would be the optimal solution?



J) What if the objective was MIN $2X - 3Y$, what would be the optimal solution?



Linear Programming Example

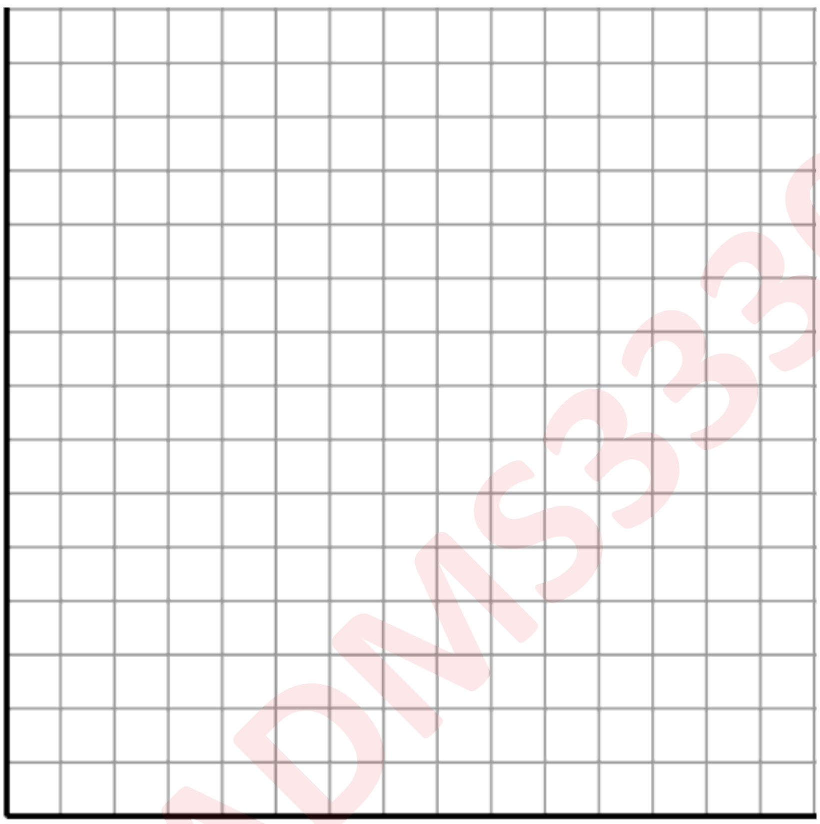
Example

Consider the following linear program.

MIN $3X + Y$
s.t.

$X - 2Y \geq 1$
$2X + Y \leq 10$
$X + Y \geq 2$
$X, Y \geq 0$

A) Graph the feasible region



B) Find the optimal solution

Linear Programming Example

C) Write the linear program in standard form.

D) Calculate the slack/surplus for each constraint.

E) What are the binding constraints?

F) Calculate the shadow prices.

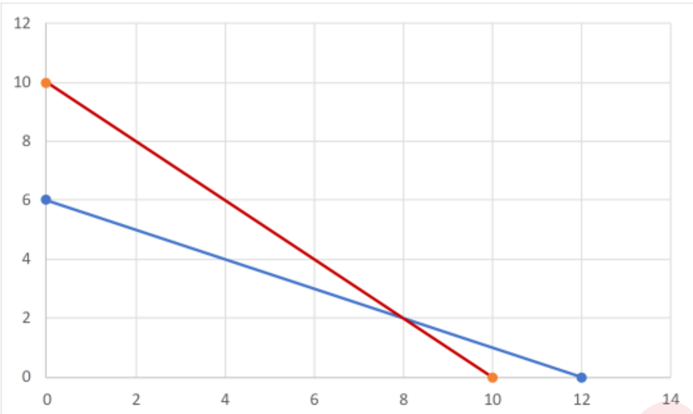
G) What if the objective was MAX 3X+Y, what would be the optimal solution?

Clarification on Objective Function Direction

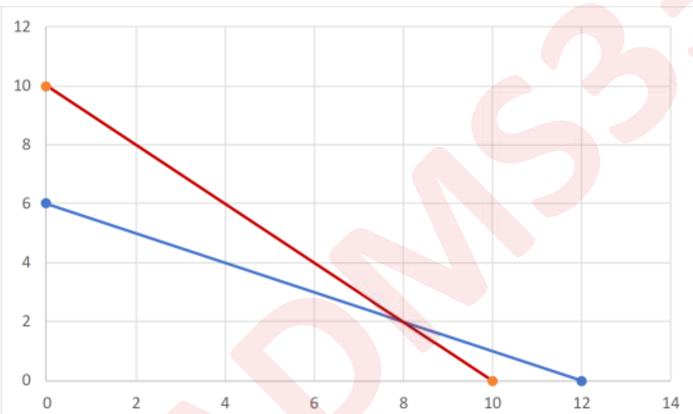
Moving Objective Function

To find which direction to move the objective function line, and be sure we are moving in the right direction, choose one point above the line, and one point below the line. (Try to choose a point along the x or y axis). Plug these points in to the objective function. The point that gives the larger value is the direction you move for maximization. The point that gives the smaller value is the direction you move for minimization.

A) MAX 2X+4Y vs MIN 2X+4Y



B) MAX 2X-4Y vs MIN 2X-4Y



C) MAX -2X+4Y vs MIN -2X+4Y

Special Cases

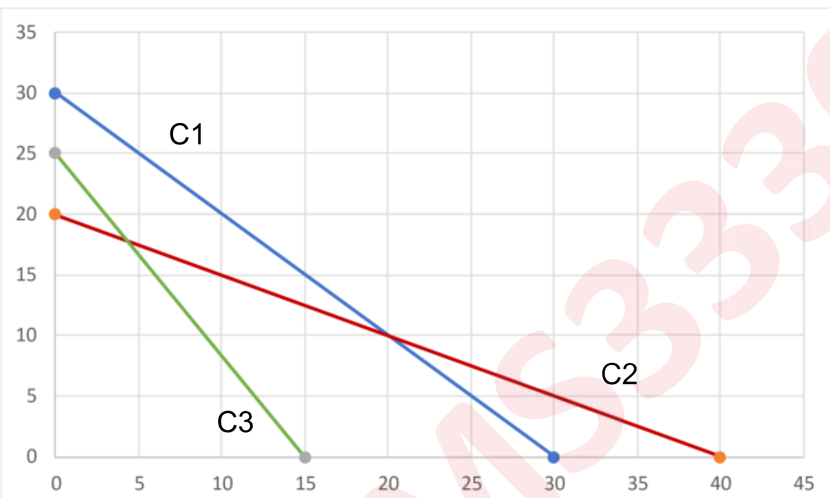
Redundant Constraint

A constraint that does not affect the feasible region and thus does not affect the optimal solution. Graphically, this is a constraint line that does not border the feasible region. If a constraint is determined to be redundant, it can be safely ignored.

Example Consider the following linear program.

MIN $5X + 6Y$
s.t.

$X + Y \geq 30$
$X + 2Y \geq 40$
$5X + 3Y \geq 75$
$X, Y \geq 0$



Special Cases

Constraints With Equality (=)

If you have a constraint with an = sign, then then solution must lie on that line.

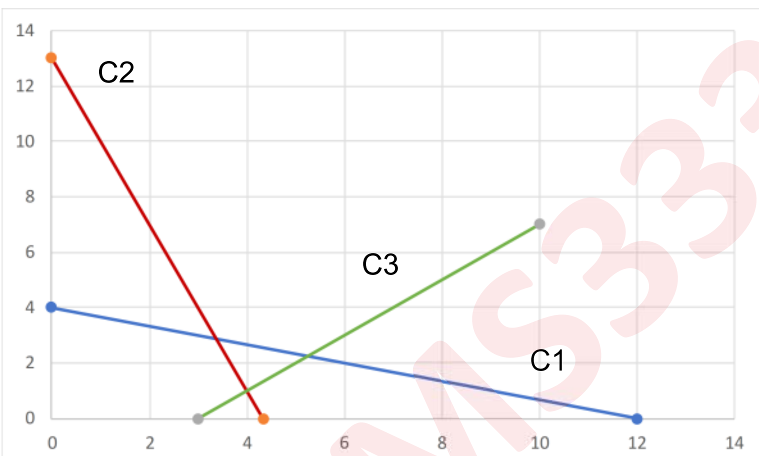
Example Consider the following linear program.

MIN $2X + 2Y$
s.t.

$X + 3Y \leq 12$
$3X + Y \geq 13$
$X - Y = 3$
$X, Y \geq 0$

A plot is provided with the constraints labeled.

A) What is the optimal solution?
B) What is the optimal solution if the objective was MAX $2X + 2Y$ instead?



Special Cases

Alternative Optimal Solutions

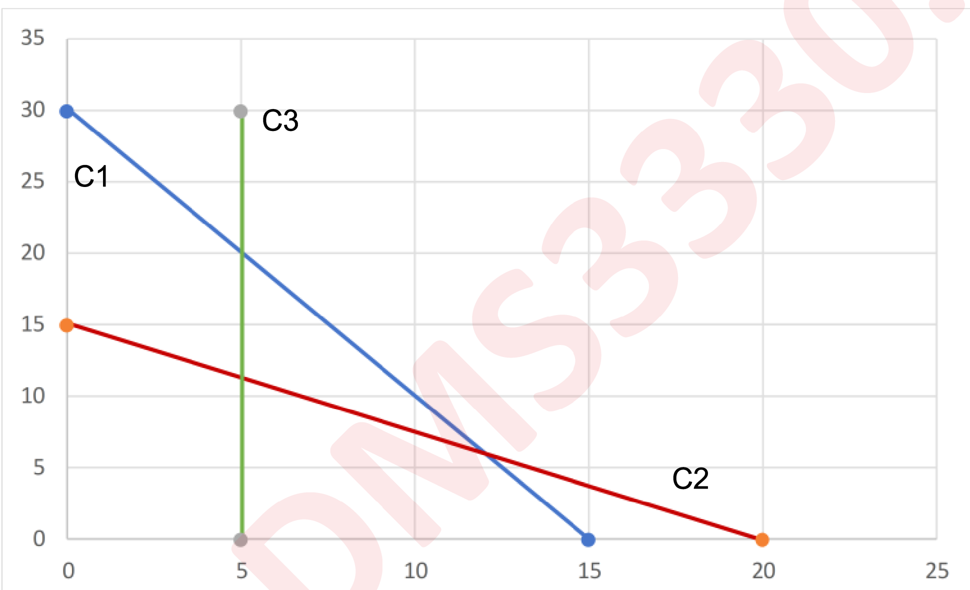
It's possible to have multiple solutions (alternative optimal solutions). This occurs when the objective function line coincides with a binding constraint line. In this case, the extreme points are a solution to the problem, as is any non-extreme point on the line connecting the extreme points.

Example Consider the following linear program.

MAX $6X + 3Y$
s.t.

$2X + Y \leq 30$
$3X + 4Y \geq 60$
$X \geq 5$
$X, Y \geq 0$

A plot is provided with the constraints labeled. What is the optimal solution?



Special Cases

Infeasibility

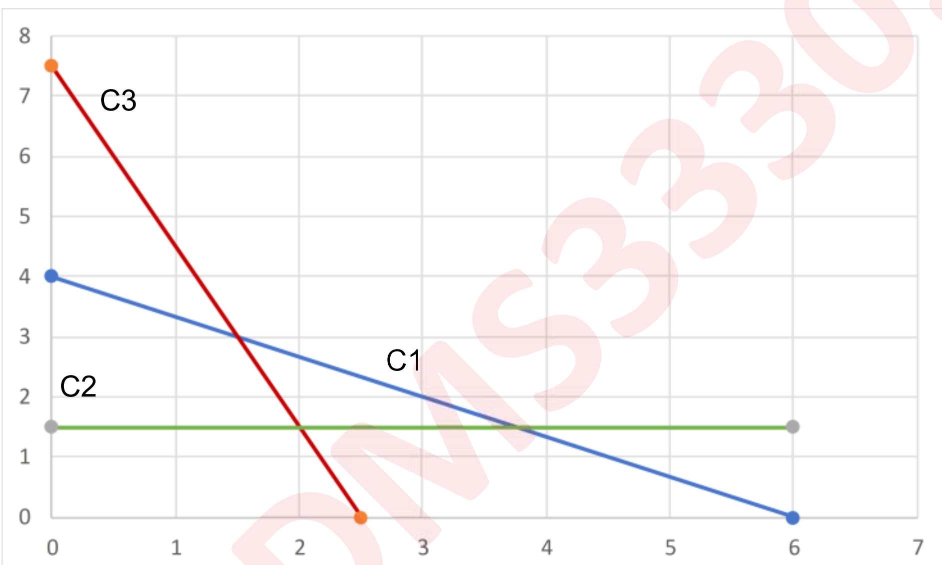
It's possible to have no solutions (infeasibility) due to no feasibility region existing. This occurs when no solution satisfies all the constraints.

Example Consider the following linear program.

MAX $25X + 40Y$
s.t.

$15X + 22.5Y \geq 90$
$120X + 40Y \leq 300$
$Y \leq 1.5$
$X, Y \geq 0$

What is the optimal solution?



Special Cases

Unboundedness

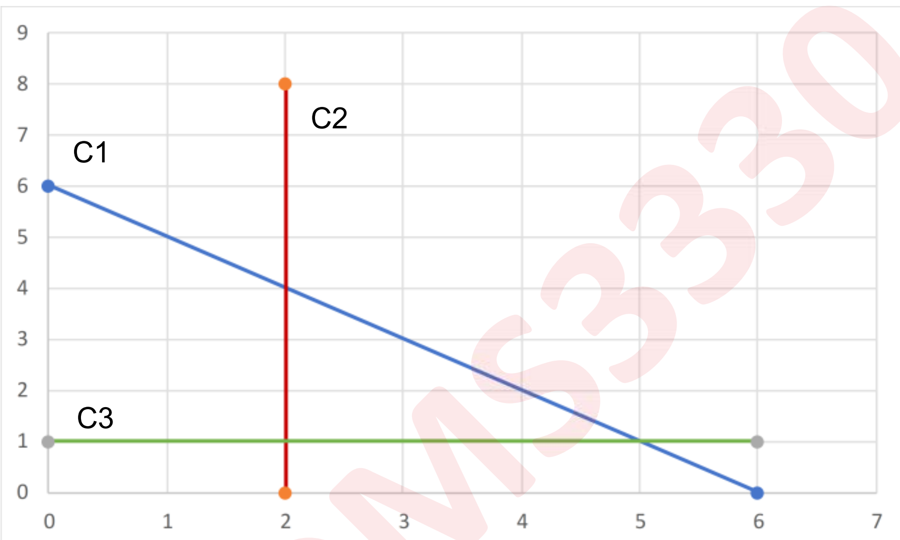
When the solution to a maximization problem is made to be infinitely large (infinitely small for a minimization problem). This typically indicates a constraint is missing or not well formed.

Example Consider the following linear program.

MAX $5X + 4Y$
s.t.

$X + Y \geq 6$
$X \geq 2$
$Y \geq 1$
$X, Y \geq 0$

What is the optimal solution?



Linear Programming - Word Problems

Example

Muir Manufacturing produces two popular grades of commercial carpeting among its many other products. In the coming production period, Muir needs to decide how many rolls of each grade should be produced in order to capitalize on their profit.

Requirements per roll of carpet are:

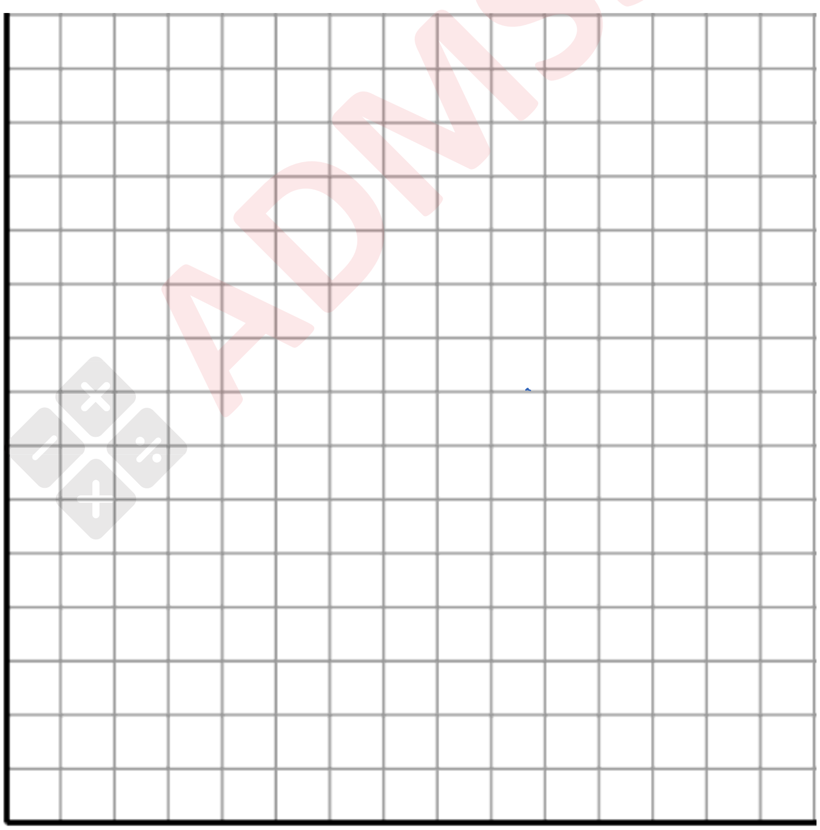
	Grade X	Grade Y
Synthetic Fiber	50 units	40 units
Production Time	25 hours	28 hours
Foam Backing	20 units	15 units

The profit per roll of Grade X carpet is $200 and the profit per roll of Grade Y carpet is $160. In the coming production period, Muir has 3000 units of synthetic fiber available for use. Workers have been scheduled to provide at least 1800 hours of production time (overtime is a possibility). The company has 1500 units of foam backing available for use.

A) Develop a linear programming model for this problem.

B) Write the model in standard form.

C) Solve the linear programming model graphically



Example

A company manufactures two products (A and B) and the profit per unit sold is \$3 and \$5 respectively. Each product has to be assembled on a particular machine, each unit of product A taking 12 minutes of assembly time and each unit of product B 25 minutes of assembly time. The company estimates that the machine used for assembly has an effective working week of only 30 hours (due to maintenance/breakdown). Technological constraints mean that for every five units of product A produced at least two units of product B must be produced.

Formulate the linear problem and solve it.